The Lifshitz–Slyozov system with inflow boundary conditions

J. Calvo[1], **E. Hingant**[2] & **R. Yvinec**[3]

1: Departamento de Matemática Aplicada and Research unit MNat, Universidad de Granada, Spain
2: Departamento de Matemática, Universidad del Bío-Bío, Chile
3: PRC, INRA, CNRS, IFCE, Cogitamus Laboratory and Université de Tours, France

Contact: juancalvo@ugr.es

1 Introduction

The Lifshitz–Slyozov system [4] describes the temporal evolution of a mixture of monomers and aggregates, where individual monomers can attach to or detach from already existing aggregates. We define:

- The **aggregate distribution** $0 \le f(t,x)$ as a function of size x and time t.
- The **monomer concentration** $0 \le u(t)$.
- The **total mass** of the system, $\rho > 0$.
- The **kinetic rates** $a(x)$ and $b(x)$, describing how fast do attachment (a given monomer attaches to a given aggregate) and detachment (a monomer detaches from a given aggregate) reactions take place.
- The relative rate $\Phi(x) = b(x)/a(x)$ and the nucleation activation threshold $\Phi_0 = \lim_{x\to 0^+} b(x)/a(x)$.
- The **nucleation rate** $\mathfrak{n}$.

The aggregate distribution follows a transport equation with respect to the size variable, whose transport rates are coupled to the dynamics of monomers through a mass conservation relation. The initial-boundary value problem for the Lifshitz–Slyozov model thus reads:

$$\begin{cases} \dfrac{\partial f(t,x)}{\partial t} + \dfrac{\partial[(a(x)u(t)-b(x))f(t,x)]}{\partial x} = 0\,, \quad t>0\,,\ x\in(0,\infty)\,, \\ u(t) + \displaystyle\int_0^\infty x f(t,x)\,dx = \rho\,, \quad t>0, \end{cases} \tag{1}$$

subject to the initial condition

$$f(0,x) = f^{in}(x)\,, \qquad x\in(0,\infty) \tag{2}$$

and the boundary condition

$$\lim_{x\to 0^+} (a(x)u(t)-b(x))f(t,x) = \mathfrak{n}(u(t))\,, \quad t>0 \tag{3}$$

whenever $u(t) > \Phi_0$. Up to date, most works on (1) utilize a set of kinetic rates $a(x)$, $b(x)$ such that the flow is outgoing at zero size and there is no need for such a boundary condition, see e.g. [2, 3].

Standard applications of the Lifshitz–Slyozov model (e.g. late stages of phase transitions) do not require a boundary condition. However, boundary effects at small sizes and nucleation phenomena become important in biologically-oriented applications: there is a growing literature on the use of this model to describe **protein polymerization** phenomena and neurodegenerative diseases. We think that new applications of this framework will gradually appear (e.g. models in Oceanography), given that the boundary term can represent the synthesis of new aggregates from monomers.

Therefore, we are interested in analyzing (1) when there holds that $u(0) > \Phi_0$, that is, when we have an inflow dynamics. To the best of our knowledge, there is no mathematical theory backing up this scenario yet. Our results below can be found in [1].

The set of assumptions

We consider kinetic rates $\{a, b, \mathfrak{n}\}$ that satisfy:

- $0 \le a,\ b \in \mathcal{C}^0([0,\infty)) \cap \mathcal{C}^1(0,\infty)\,,$
- a' and b' are bounded on $(1,\infty)\,,$
- $a(x) > 0$ for all $x > 0$ and $\frac{1}{a} \in L^1(0,1)\,,$
- $\Phi' \in L^1(0,1)\,,$
- $0 \le \mathfrak{n}$ is locally Lipschitz on $[\Phi_0,\infty)\,.$

We restrict the choice of initial data to

- $f^{\text{in}} \in L^1((0,\infty),(1+x)\,dx)\,,$
- $u^{\text{in}} := \rho - \int_0^\infty x f^{\text{in}} > \Phi_0\,,$

so that the balance of mass in (1) makes sense at time $t=0$ and the flow has incoming character.

2 Existence of local solutions

We prove existence by a fixed point argument on $u(t)$, through a map G defined on $\{u \mid u : [0,T) \to [0,\rho] \text{ continuous}\}$:

$$u(t) \mapsto f(t,x) \mapsto \tilde{u}(t) = G(u)(t) = \left(\rho - \int_0^\infty x f(t,x)\,dx\right)$$

where $f(t,x)$ solves the transport equation in (1) with the given $u(t)$.

Characteristics and mild solutions

We write $v(t,x) := a(x)u(t) - b(x)$ for the transport field and define the associated characteristics as the maximal solutions of

$$\begin{aligned} \frac{dX(s;t,x)}{dt} &= v(s, X(s;t,x))\,, \\ X(t;t,x) &= x, \end{aligned} \tag{4}$$

defined on the time interval $J_{t,x} \subset [0,\infty)$. The associated Jacobian is

$$J(s;t,x) = \exp\left(-\int_s^t (\partial_x v)(\tau, X(\tau;t,x))\,d\tau\right)\,.$$

Due to the incoming flux, we also need to track trajectories stemming from $x=0$. We define the *enter-time* associated to the characteristic curve passing through (t,x) as

$$\sigma_t(x) := \inf J_{t,x}$$

(time at which the curve $s \mapsto (s, X(s;t,x))$ enters phase space). Introducing the separating point

$$x_c(t) := \inf\{x > 0 \mid \sigma_t(x) = 0\}$$

we partition size space $(0,\infty)$ as follows:

Proposition 2.1. *For each $t \in (0,T)$, the following holds:*

- *The map $x \mapsto X(t;0,x)$ is an increasing $\mathcal{C}^1$-diffeomorphism from $(0,\infty)$ to $(x_c(t),\infty)$. Moreover, we have that $\lim_{x\to 0^+} X(t;0,x) = x_c(t)$.*
- *The map $s \mapsto \sigma_t^{-1}(s)$ is a decreasing $\mathcal{C}^1$-diffeomorphism from $(0,t)$ to $(0,x_c(t))$. Moreover, $\sigma_t^{-1}(s) = \lim_{x\to 0^+} X(t;s,x)$.*

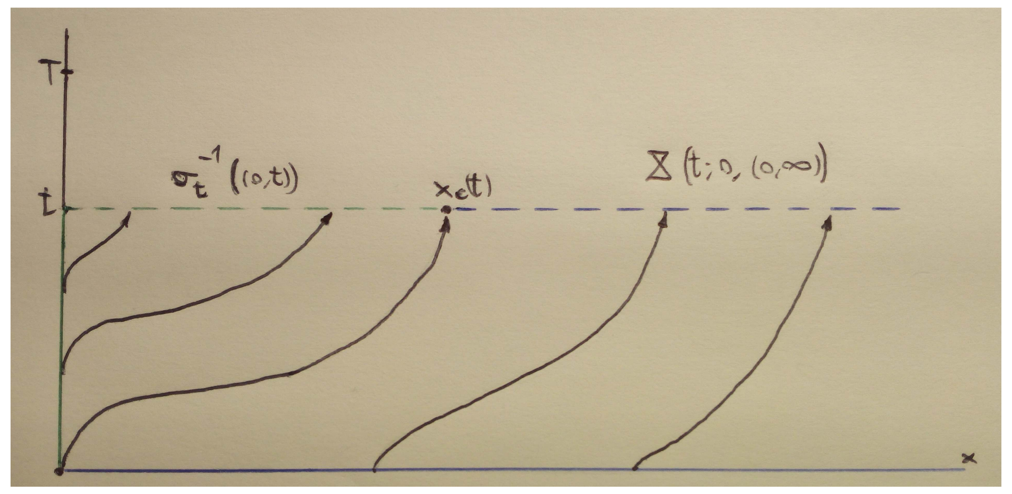

Figure 1: The size domain $(0,\infty)$ is partitioned at time t as the union of $\sigma_t^{-1}((0,t))$ and $X(t;0,(0,\infty))$, according to whether the characteristics reaching a given size x came from the boundary or the interior of the domain.

All these notions allow to represent solutions as follows:

$$\begin{aligned} f(t,x) = {} & f^{\text{in}}(X(0;t,x))J(0;t,x)\mathbf{1}_{(x_c(t),\infty)}(x) \\ & + \mathfrak{n}(u(\sigma_t(x)))|\sigma_t'(x)|\mathbf{1}_{(0,x_c(t))}(x)\,. \end{aligned} \tag{5}$$

Fixed point argument

The regularity of characteristic trajectories entails the continuity of the fixed-point map G. Then Schauder's fixed point theorem can be applied and we obtain:

Theorem 2.2 (Existence of solution)**.** *Let $\{a,b,\mathfrak{n}\}$, $\rho > \Phi_0$ and f^{in} satisfy our hypotheses. Then, there exists at least one weak solution $0 \le f \in \mathcal{C}([0,T); w - L^1((0,\infty),(1+x)\,dx))$ to the Lifshitz–Slyozov initial-boundary value problem* (1)–(2)–(3)*. This solution is such that:*

- *For each $T^* < T$, f belongs to $L^\infty((0,T^*), L^1((0,\infty),dx))$.*
- *It can be represented in terms of characteristics, cf. formula* (5) *above.*

Moreover, if T is taken to be maximal then either $T=\infty$ or $T<\infty$ and $u(t) \to \Phi_0$ as $t \to T$.

3 Uniqueness

The solution constructed in Theorem 2.2 can be shown to be unique under some additional assumptions.

Theorem 3.1 (Uniqueness of solution)**.** *Assume that there exists $x^* > 0$ such that Φ is monotone on $[0,x^*)$. Then, for any initial data such that $f^{\text{in}} \in L^1((0,\infty),(1+x+x^2)\,dx)$ and for all $T>0$, there exists at most one weak solution to* (1)–(2)–(3) *on* $(0,T)$.

The proof of this statement is obtained by an adaptation of the technique introduced in [3]: we derive an evolution equation for quantities of the form

$$F^+(t,x) = \int_x^\infty f(t,y)\,dy\,.$$

Gronwall estimates allow to show uniqueness for these quantities.

4 Local and global solutions

The maximality statement in Theorem 2.2 above is optimal, in the sense that there are examples of both global solutions and local-in-time solutions that cannot be extended further.

Proposition 4.1. *Assume $\Phi(x) \ge \Phi_0$ for all $x > 0$. Then, the solution to the Lifshitz–Slyozov equation constructed in Theorem 2.2 is global, that is $T = \infty$.*

Proposition 4.2. *Assume that f^{in} is compactly supported, that Φ is convex and strictly decreasing and that there exists numbers $\underline{a}$, $\overline{a}$ such that $0 < \underline{a} < a(x) < \overline{a} < \infty$ for all $x > 0$. Then, the solution to the Lifshitz–Slyozov equation constructed in Theorem 2.2 is not global, that is, u reaches Φ_0 in finite time.*

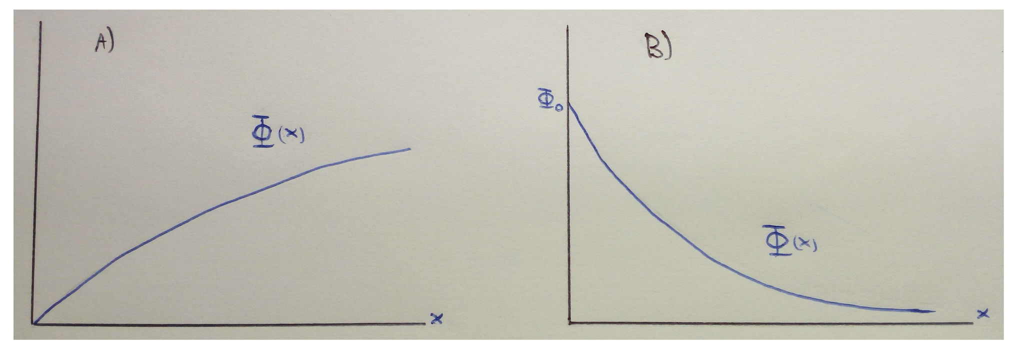

Figure 2: A) Example of a relative rate for which solutions are global. B) Example of a relative rate for which solutions are local.

Acknowledgements

J. C. acknowledges support from MICINN, projects MTM2017-91054-EXP and RTI2018-098850-B-IOO; he also acknowledges support from Plan Propio de Investigación, Universidad de Granada, Programa 9 -partially through FEDER (ERDF) funds-. E. H. acknowledges support from FONDECYT Iniciación n° 11170655. R. Part of this work was done while J. C. and R. Y. were visiting the Departamento de Matemática at Universidad del Bío-Bío and while E. H. and J. C. were visiting Institut Denis Poisson at Université de Tours and INRA Tours.

References

[1] J. Calvo, E. Hingant, R. Yvinec, Initial-boundary value problem to the Lifshitz–Slyozov equation with non-smooth rates at the boundary. ArXiv preprint 2004:01947.

[2] J.-F. Collet and T. Goudon. On solutions of the Lifshitz-Slyozov model. *Nonlinearity*, 13(4):1239–1262, 2000.

[3] P. Laurençot. Weak solutions to the Lifshitz-Slyozov-Wagner equation. *Indiana Univ. Math. J.*, 50(3):1319–1346, 2001.

[4] I. M. Lifshitz and V. V. Slyozov. The kinetics of precipitation from supersaturated solid solutions. *J. Phys. Chem. Solids*, 19(1-2):35–50, 1961.